Several iterative algorithms for solving the elasticity equation are described by the finite element method are theoretically scalable in the sense that the number of iterations does not grow much when the mesh is refined for better accuracy or when the number of processors is increased. However, the theoretically optimal scalability does not translate into linear scalability in terms of the computer time. We introduce a new way to construct coarse level spaces that preserve the geometric structure of the computational domain, but give up accuracy in the interior of the domain. As it turns out the tradeoff in accuracy promises the high scalability in terms of the total computer time. The low accuracy coarse space does not change the overall accuracy once it is part of the preconditioner. We show numerically that such a new preconditioner is highly scalable for solving linear elasticity equations discretized on unstructured 3D meshes with hundreds of millions of unknowns on a supercomputer with over 10,000 processors.

### Linear elasticity equation

The following linear elasticity equation [3] is used to calculate the displacement $u$ of a body $u(D)$ which is fixed along a portion of its boundary, $\Gamma_D$, and is subject to a surface force $f$ along the rest of the boundary $\Gamma_N = \partial \Omega \setminus \Gamma_D$. The variational formulation of (1) reads as

$$\int_\Omega \nabla u \cdot (\mu \nabla v) \, dx + \int_{\Gamma_N} g v \, ds = \int_D f v \, dx$$

with $u \in H^1(\Omega)$ and $v \in H^1_0(\Omega)$. Here $\mu$ is the shear modulus, $g$ is the surface tension, $\Gamma_N$ is the outward unit normal to $\partial \Omega$, $\mu$ and $\lambda$ are lame's coefficients expressed as functions of Young's modulus, $E$, and Poisson's ratio, $\nu$, by

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

The variational formulation of (1) reads as

$$\int_\Omega \nabla u \cdot (\mu \nabla v) \, dx = \int_D f v \, dx$$

for all $v \in H^1_0(\Omega)$, such that

$$(\varepsilon(u), \varepsilon(v)) + 2\mu (\sigma(u), \sigma(v)) = (f, v)$$

for all $v \in H^1_0(\Omega)$, where $\varepsilon(u) = (1/2)(\nabla u + \nabla u^T)$, $\sigma(u) = \mu \varepsilon(u) - \lambda \text{tr} \varepsilon(u)$, and $\text{tr} \varepsilon(u) = 0$.

### Three-level restricted Schwarz preconditioner

Three-level restricted Schwarz preconditioner is defined as a restriction $\sigma_a$ that returns the vector of coefficients defined in the interior of $\Omega_a$, $\sigma_a(u) = \sigma_a(u)|_{\Omega_a}$, as a restriction that returns the vector of coefficients defined in the interior of $\Omega_{a\Gamma_1}$, $\sigma_{a\Gamma_1}(u) = \sigma_{a\Gamma_1}(u)|_{\Omega_{a\Gamma_1}}$, and as a restriction that returns the vector of coefficients defined in the interior of $\Omega_{a\Gamma_1\Gamma_2}$, $\sigma_{a\Gamma_1\Gamma_2}(u) = \sigma_{a\Gamma_1\Gamma_2}(u)|_{\Omega_{a\Gamma_1\Gamma_2}}$, respectively.

For convenience, three meshes from fine to coarse are denoted by $h_0$, $h_1$, and $h_2$. The restricted Schwarz preconditioner on the level $h_i$ is defined as

$$P^i = \frac{1}{h_i^2} P^{i-1} + R^i$$

where $P^{i-1}$ is the restricted Schwarz preconditioner on the level $h_{i-1}$ and $R^i$ is some restriction operator. The three-level method can be written as:

$$P^2 = \frac{1}{h_2^2} P^{1} + R^2$$

or

$$P^2 = \frac{1}{h_2^2} (P^1 + h_2^2 R^2)$$

where

$$P^1 = \frac{1}{h_1^2} (P^0 + h_1^2 R^1)$$

and $P^0 = I$. After the convergence is achieved, the meshes are refined and the iteration is repeated.

### Numerical results

We consider a three-dimensional lever, which is part of a dental CT image, as shown in Fig 2, where a loading of 5000 ($\text{N/m}^2$) is applied to the lever at the free end and the last cylinder is all fixed. The Young's modulus $E$ and Poisson ratio $\nu$ are assigned as $2.15 \times 10^5 \text{Pa}$ and 0.29 respectively. The algorithms are implemented on top of the PETSc [4] and the numerical experiments are carried on a cluster of IBM servers.

References

[1] X.-C. Cai, M. Sarkis, A restricted additive Schwarz preconditioner for general sparse linear systems with more than 260 millions unknowns, and on a machine with over 10,000 processors.


